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ABSTRACT

We give an introduction to the theory of linear systems as put forward by Willems. The approach is through polynomial algebra.

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0. INTRODUCTION

In recent years J. C. Willems has developed a new approach to finite-dimensional linear systems theory [1, 2]. This paper is meant to be a more or less tutorial introduction to parts of this new approach for readers with a bias toward linear algebra. We will demonstrate how far one can come by using a few tools from polynomial linear algebra. It is clear that without Willems's seminal work this paper would not have been written. For historical reasons, and reasons of honesty, we indicate the main novelties of this paper. The proofs of Theorems 9 and 10 are new, in the sense that they are not printed in the open literature elsewhere. Sections 11 and 12 are new. Section 13 contains some new elements in the proofs. Section 17's approach differs somewhat from existing approaches. Section 18 contains a new proof of an existing result. Section 19 is new. The sections about control (21, 22, and 23) contain new material. But even for the more or less new parts it holds true that without the very close collaboration with Willems they would not have been written. In this sense this paper should be considered as an intellectual homage to J. C. Willems.

We have tried to do our utmost to make this paper self-contained, in the sense that all major theorems from polynomial linear algebra not known to

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everyone are written down (without proof). Most systems-theoretic results are given *with* proof, the major exception being some parts about realization and state-space theory.

NOTATION used throughout the whole paper. $\mathbb{R} :=$ reals, $\mathbb{Z} :=$ all integers, $\mathbb{Z}_+ := \{z \in \mathbb{Z} \mid z \geqslant 0\}$, $\mathbb{N} :=$ natural numbers. With $q \in \mathbb{Z}_+$ we denote by $\mathbb{R}^q$ the q-dimensional Euclidean space. Vectors in $\mathbb{R}^q$ are supposed to be column vectors. $(\mathbb{R}^q)^{\mathbb{Z}} := \{w : \mathbb{Z} \to \mathbb{R}^q\}$. Let $\tau \in \mathbb{Z}$; then $\sigma^\tau : (\mathbb{R}^q)^{\mathbb{Z}} \to (\mathbb{R}^q)^{\mathbb{Z}}$ is defined by $(\sigma^\tau w)(t) := w(t+\tau)$ $\forall t \in \mathbb{Z}$, $\forall w \in (\mathbb{R}^q)^{\mathbb{Z}}$. We call σ the *backward shift*. $\mathfrak{L}^q := (\mathbb{R}^q)^{\mathbb{Z}}$ with the topology of pointwise convergence; see Section 1 for a definition.

Let $n_1, n_2 \in \mathbb{N}$; then $\mathbb{R}^{n_1 \times n_2}[s, s^{-1}] := \{$matrices M with n_1 rows and n_2 columns such that every entry of M is a polynomial in s and $s^{-1}\}$. See also Section 2.

The *rank* of a polynomial matrix is meant to be its rank over the *rational* functions.

1. TOPOLOGICAL PRELIMINARIES

Let $\langle \ , \ \rangle$ denote the standard inner product on $\mathbb{R}^q$. Let $W := (\mathbb{R}^q)^{\mathbb{Z}}$ and $W^* := \{w \in W \mid \exists t_-(w), t_+(w) \in \mathbb{Z}$ such that $w(t) = 0$ $\forall t < t_-(w)$, $\forall t > t_+(w)\}$. An element from W (W^*) will be denoted by w (w^*). We define the following bilinear form on $W \times W^*$: $\langle w, w^* \rangle := \sum_{t \in \mathbb{Z}} \langle w(t), w^*(t) \rangle$.

With this bilinear form, (W, W^*) is a so-called *dual pair*; see [2]. We endow W (W^*) with the coarsest topology such that the linear forms in W^* (W) are continuous. Hence a base of (closed) neighborhoods of the origin in W (W^*) is formed by sets of the form $\{w \mid \sup_{1 \leqslant i \leqslant n} |\langle w_i^*, w \rangle| \leqslant 1\}$, where $n \in \mathbb{N}$ is arbitrary and $w_i^* \in W^*$ $\forall i \in \{1,2,\ldots,n\}$ ($\{w^* \mid \sup_{i \leqslant i \leqslant n} |\langle w^*, w_i \rangle| \leqslant 1\}$, where $w_i \in W$ $\forall i \in \{1,2,\ldots,n\}$).

NOTATION. $\mathfrak{L}^q := W$ with the topology defined above.
$\mathfrak{L}_c^q(d) := W^*$ with the topology defined above.
As a mnemonic device, c stands for *compact* and d for *dual*.

DEFINITION. $\mathfrak{L}_c^q(p) := W^*$ with the relative topology of $\mathfrak{L}^q$. Here the p stands for *primal*.

One can prove that $(\mathfrak{L}^q)^* := \{$all linear continuous forms on $\mathfrak{L}^q\} = W^*$, and $[\mathfrak{L}_c^q(d)]^* = W$; see [3]. One can further prove that $\mathfrak{L}^q$ is a separable metrizable complete convex space, i.e., a Fréchet space. Although $\mathfrak{L}_c^q(d)$ is a convex space, it has not a countable base of neighborhoods of the origin. As

$\mathfrak{L}^q$ is a Frêchet space, its topology is completely specified by specifying all convergent *sequences* in $\mathfrak{L}^q$.

One can prove that $\{w_i,\ w \in \mathfrak{L}^q,\ w_i \to_{i\to\infty} w \ \Leftrightarrow\ w_i(t) \to_{i\to\infty} w(t)\ \forall t \in \mathbb{Z}\}$. Hence one can say that $\mathfrak{L}^q$ is W endowed with the *topology* of *pointwise convergence*.

Although the topology of $\mathfrak{L}_c^q(d)$ is not completely specified by sequences, one can nevertheless prove the following:

If $w_i^*, w^* \in \mathfrak{L}_c^q(d)$, then $w_i^* \to_{i\to\infty} w^* \ \Leftrightarrow\ \exists t_-, t_+ \in \mathbb{Z}$ such that for i sufficiently large, $w_i^*(t) = 0\ \forall t < t_-$, $\forall t > t_+$ and $w_i^*(t) \to_{i\to\infty} w^*(t)\ \forall t \in \mathbb{Z}$.

One can also consider $\mathfrak{L}^q$ as the *projective limit* of the Euclidean spaces $\mathbb{R}^q$, and $\mathfrak{L}_c^q(d)$ as the *inductive limit* of the spaces $\mathbb{R}^q$. In infinite-dimensional systems theory one also uses the construction implicit in the notions of inductive and projective limits; see [4].

2. ALGEBRAIC PRELIMINARIES

Consider in W^* the following elements e_i defined by $e_i(t) = \varepsilon_i$ when $t = 0$, $e_i(t) = 0$ when $t \neq 0$, where ε_i is the ith unit row vector in $\mathbb{R}^q$. It is now evident that every element $w^* \in W^*$ can be written as $\sum_{i=1}^q p_i(\sigma, \sigma^{-1})e_i$ where $p_i(\sigma, \sigma^{-1})$ are polynomials in σ and σ^{-1} (the forward and the backward shift). Hence we can identify W^* with $\mathbb{R}^q[s, s^{-1}]$, the set of *polynomials* $\sum_{t\in\mathbb{Z}} a_t s^t$ with $a_t \in \mathbb{R}^q\ \forall t$.

In accordance with the usual componentwise addition in W^*, we define addition componentwise in $\mathbb{R}^q[s, s^{-1}]$, i.e., with $a(s) = \sum a_t s^t$ and $b(s) = \sum b_t s^t$, we define $c(s) := a(s) + b(s)$ by $c(s) = \sum c_t b^t$ and $c_t = a_t + b_t\ \forall t \in \mathbb{Z}$. In case $q = 1$ we also define multiplication in $\mathbb{R}[s, s^{-1}]$ as follows: Let $a(s)$ and $b(s)$ be as above (with $a_t, b_t \in \mathbb{R}\ \forall t$); then $d(s) := a(s)b(s)$ is defined by $d(s) = \sum d_t s^t$ and $d_t = \sum_{t'+t''=t} a_{t'} b_{t''}$. As elements in $\mathbb{R}[s, s^{-1}]$ have also a finite number of nonzero terms, this multiplication is well defined. We turn $\mathbb{R}^q[s, s^{-1}]$ into a module over the ring $\mathbb{R}[s, s^{-1}]$ by means of the following "scalar" multiplication: Take $\alpha(s) := \sum \alpha_t s^t \in \mathbb{R}[s, s^{-1}]$ and $b(s) := \sum b_t s^t \in \mathbb{R}^q[s, s^{-1}]$; then $\alpha(s)\cdot b(s) = \sum c_t s^t$, where $c_t := \sum_{t'+t''=t} \alpha_{t'} b_{t''}$. Let $\alpha(s) = \sum \alpha_t s^t \in \mathbb{R}[s, s^{-1}]$; then $\mathfrak{L}_+ := \max\{t \mid \alpha_t \neq 0\}$ and $\mathfrak{L}_- := \min\{t \mid \alpha_t \neq 0\}$. Define $\gamma(\alpha(s)) := |\mathfrak{L}_+ - \mathfrak{L}_-|$; then it is not hard to show that γ is a *degree function* on $\mathbb{R}[s, s^{-1}]$. One can even show that with this degree function $\mathbb{R}[s, s^{-1}]$ is a *Euclidean domain*. Using a famous theorem by Hilbert (see for instance [8]), one can prove that every submodule of $\mathbb{R}^q[s, s^{-1}]$ (over the ring $\mathbb{R}[s, s^{-1}]$) is finitely generated. By means of simple polynomial operations one can even prove that the dimension of submodules in $\mathbb{R}^q[s, s^{-1}]$ is at most q.

NOTATION. Let $r_1(s),\dots,r_g(s)$ be row vectors in $\mathbb{R}^q[s,s^{-1}]$, and $R(s) := \operatorname{col}(r_1(s),\dots,r_g(s))$. Then $[\![r_1(s),\dots,r_g(s)]\!] := [\![R(s)]\!] :=$ module generated by $\{r_1(s),\dots,r_g(s)\}$.

Let $U(s,s^{-1}) \in \mathbb{R}^{g\times g}[s,s^{-1}]$ be an arbitrary unimodular matrix, i.e., $U(s,s^{-1})^{-1} \in \mathbb{R}^{g\times g}[s,s^{-1}]$. Then it is not hard to prove that $[\![R(s)]\!] = [\![U(s,s^{-1})R(s)]\!]$.

Take g such that rank $R(s) = g =$ number of basis elements in $[\![R(s)]\!] =$ rank $R(s)$ over the rational functions $=$ rank $R(c)$ for almost all $c \in \mathbb{C}$. Then one can show by means of a *constructive* proof that there is a unimodular matrix $U(s,s^{-1})$ such that $\hat{R}(s) := U(s,s^{-1})R(s)$ is *bilaterally row proper*, i.e., $\hat{R}(s) = \hat{R}_0 + \hat{R}_1 s + \cdots + \hat{R}_\delta s^\delta$, $\hat{R}_i \in \mathbb{R}^{g\times q}$, $\delta \geqslant 0$. If $\hat{R}(s) = \operatorname{diag}(s^{\delta_1},\dots,s^{\delta_g})\hat{R}_h + \hat{R}_{\text{rest}}$, one has rank $\hat{R}_0 =$ rank $\hat{R}_h = g$ ($\hat{R}_h \in \mathbb{R}^{g\times q}$). Of course $\delta = \max\{\delta_1,\dots,\delta_g\}$. So every submodule in $\mathbb{R}^q[s,s^{-1}]$ can be given a bilaterally row proper basis.

EXAMPLE. Take

$$R(s) := \begin{pmatrix} 1+s & 0 & 0 \\ 0 & 1 & s^2 \end{pmatrix}.$$

Then

$$\hat{R}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \hat{R}_h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

hence $R(s)$ is bilaterally row proper.

In the rest of this paper bilaterally row proper matrices play an important role.

3. INTRODUCTION OF $\underline{\mathfrak{L}}^q$

Central in this paper is the set of all linear, shift-invariant, and closed subspaces of $\mathfrak{L}^q$. To be precise:

DEFINITION. $\underline{\mathfrak{L}}^q := \{\mathfrak{B} \subseteq \mathfrak{L}^q \mid 1, 2, 3 \text{ below hold}\}$:

1. $\mathfrak{B}$ is a linear subspace of $(\mathbb{R}^q)^{\mathbb{Z}}$.
2. $\mathfrak{B}$ is closed.
3. $\mathfrak{B}$ is shift-invariant, i.e., $\sigma\mathfrak{B} = \mathfrak{B}$.

NOTATION.

$$\mathfrak{B}[t_1, t_2] := \{x \in \mathbb{R}(t_2 - t_1 + 1)q \mid \exists w \in \mathfrak{B} \text{ with } x = (w(t_1), w(t_1+1), \dots, w(t_2))\}.$$

DEFINITION. $\mathfrak{B} \subset (\mathbb{R}^q)^{\mathbb{Z}}$ is called *complete* if $\{(w(t_1), \dots, w(t_2)) \in \mathfrak{B}[t_1, t_2] \ \forall t_1 \leqslant t_2 \in \mathbb{Z}\} \Leftrightarrow w \in \mathfrak{B}$.

Hence $\mathfrak{B}$ is complete if membership of $\mathfrak{B}$ is completely determined by information about finite parts. One can prove the following

THEOREM (see [1]). *$\mathfrak{B}$ is closed in $\mathfrak{L}^q$ iff $\mathfrak{B}$ is complete.*

4. AUTOREGRESSIVE REPRESENTATIONS OF ELEMENTS FROM $\underline{\mathfrak{L}}^q$

THEOREM [1].

1. *$\mathfrak{B} \in \underline{\mathfrak{L}}^q$ iff there is a polynomial matrix $R(s) \in \mathbb{R}^{g \times q}[s, s^{-1}]$ with $g \leqslant q$ such that $\mathfrak{B} = \{w \mid (R(\sigma)w)(t) = 0 \ \forall t \in \mathbb{Z}\} = \{w \mid R(\sigma)w = 0\}$.*
2. *Let $\mathfrak{B}_i \in \underline{\mathfrak{L}}^q$ be defined by $\mathfrak{B}_i = \{w \mid R_i(\sigma)w = 0\}$, where $R_i(s) \in \mathbb{R}^{g_i \times q}[s, s^{-1}]$, $i = 1, 2$. Then $\mathfrak{B}_1 \subseteq \mathfrak{B}_2$ iff $\mathfrak{B}_2^{\perp} \subseteq \mathfrak{B}_1^{\perp}$ iff $\exists V(s) \in \mathbb{R}^{g_2 \times g_1}[s, s^{-1}]$ such that $R_2(s) = V(s)R_1(s)$.*

Proof. 1: Let $\mathfrak{B}^{\perp} := \{w^* \in \mathfrak{L}_c^q(d) \mid \langle w^*, w\rangle = 0 \ \ \forall w \in \mathfrak{B}\}$. Then, almost by definition of $\mathfrak{L}_c^q(d)$, $\mathfrak{B}^{\perp}$ is a closed linear subset of $\mathfrak{L}_c^q(d)$. As $\mathfrak{L}_c^q(d)$ can be identified with the module $\mathbb{R}^q[s, s^{-1}]$ over the ring $\mathbb{R}[s, s^{-1}]$, it follows (see Section 2) that $\mathfrak{B}^{\perp}$ is a finitely generated submodule of $\mathfrak{L}_c^q(d)$, i.e., there is a polynomial matrix $R^T(s^{-1}) \in \mathbb{R}^{q \times g}[s, s^{-1}]$ such that $\mathfrak{B}^{\perp} = \{w^* \in \mathfrak{L}_c^q(d) \mid w^* = R^T(\sigma^{-1})\xi^* \text{ for some } \xi^* \in \mathfrak{L}_c^g(d)\}$. Hence it is evident that $R(\sigma)w = 0$, $\forall w \in \mathfrak{B}$.

Assume now, to the contrary, that for some $\hat{w} \notin \mathfrak{B}$ we have $R(\sigma)\hat{w} = 0$. As $\mathfrak{L}^q$ is locally convex, the Hahn-Banach separation theorem implies the existence of an element $w^* \in \mathfrak{L}_c^q(d)$ such that $w^* \in \mathfrak{B}^{\perp}$ and $\langle w^*, \hat{w}\rangle \neq 0$. But $w^* \in \mathfrak{B}^{\perp}$ iff $w^* = R^T(\sigma^{-1})\xi^*$ and $\langle \hat{w}, R^T(\sigma^{-1})\xi^*\rangle = \langle R(\sigma)\hat{w}, \xi^*\rangle$, and a contradiction arises.

2: Assume that $\mathfrak{B}_1 \subseteq \mathfrak{B}_2$. Then it is evident that $\mathfrak{B}_2^{\perp} \subseteq \mathfrak{B}_1^{\perp}$. As $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are closed and linear, it also follows that $\mathfrak{B}_1 \subseteq \mathfrak{B}_2$ implies $\mathfrak{B}_2^{\perp} \subseteq \mathfrak{B}_1^{\perp}$. Assume now that $R_2 = VR_1$; then evidently $\mathfrak{B}_1 \subseteq \mathfrak{B}_2$. To conclude the proof, assume that $\mathfrak{B}_1 \subseteq \mathfrak{B}_2$. Then, of course $U(\sigma)\mathfrak{B}_1 \subseteq U(\sigma)\mathfrak{B}_2$ and vice versa for every unimodular matrix $U(s) \in \mathbb{R}^{q \times q}[s, s^{-1}]$. This implies that we can reduce the proof to the case of a diagonal matrix $R_1(s)$; see Lemma 2 in the next section. It is now not hard to see that every row from $R_2(s)$ is a polynomial combination of rows from $R_1(s)$, and we are done with the proof. ■

REMARK. The second part of this result appears to be very helpful in proving various results concerning $\underline{\mathfrak{L}}^q$. In the context of *duality* we will discuss in more detail the relations between $\mathfrak{B}$, $\mathfrak{B}^\perp$, and cl $\mathfrak{B}^\perp$ (the closure of $\mathfrak{B}^\perp$) in the topology of pointwise convergence. From Theorem 2 we deduce the following. Let $\mathfrak{B} \in \underline{\mathfrak{L}}^q$ be such that $\mathfrak{B} = \{w \mid R_i(\sigma)w = 0\}$, $R_i(s) \in \mathbb{R}^{g_i \times q}[s, s^{-1}]$, rank $R_i(s) = g_i$, $i = 1,2$. Then $g_1 = g_2 =: g(\mathfrak{B})$. Later on we give an interpretation of the invariant $g(\mathfrak{B})$. ■

5. AUTOREGRESSIVE-MOVING-AVERAGE REPRESENTATIONS

DEFINITION. A subset $\mathfrak{B} \subseteq (\mathbb{R}^q)^{\mathbf{Z}}$ is said to have an *autoregressive-moving-average* (ARMA) representation if $\exists g, p \in \mathbb{Z}_+$, $\exists R(s) \in \mathbb{R}^{g \times q}[s, s^{-1}]$, $\exists M(s) \in \mathbb{R}^{g \times p}[s, s^{-1}]$ such that $\mathfrak{B} = \{w \in (\mathbb{R}^q)^{\mathbf{Z}} \mid \exists a \in (\mathbb{R}^p)^{\mathbf{Z}}$ with $R(\sigma)w = M(\sigma)a\}$.

In modeling q signals it often appears to be useful to add, say, p auxiliary signals in order to describe the laws governing $w \in (\mathbb{R}^q)^{\mathbf{Z}}$. As a mnemonic device, the letter a in the ARMA representation stands for *auxiliary*.

In the sequel we will prove that an ARMA model is an element of $\underline{\mathfrak{L}}^q$. In order to do that we need two lemmas.

LEMMA 1 [1]. *Let $0 \neq p(s) \in \mathbb{R}[s]$. Then $p(\sigma): \mathbb{R}^{\mathbf{Z}} \to \mathbb{R}^{\mathbf{Z}}$ is a surjective map.*

LEMMA 2 [5]. *Let $R(s) \in \mathbb{R}^{q \times q}[s]$. Then there are unimodular matrices $U(s) \in \mathbb{R}^{g \times g}[s]$ and $V(s) \in \mathbb{R}^{q \times q}[s]$ such that $\Delta(s) := U(s)R(s)V(s)$ is diagonal.*

Actually one can prove even more, and this leads to the so-called Smith form of $R(s)$.

THEOREM [2]. *Let $\mathfrak{B} = \{w \mid \exists a$ with $R(\sigma)w = M(\sigma)a\}$. Then $\mathfrak{B} \in \underline{\mathfrak{L}}^q$.*

Proof. (Constructs an AR representation of $\mathfrak{B}$.) Take $U(s)$ and $V(s)$ such that $U(s)M(s)V(s) =: \Delta(s)$ is diagonal and $U(s)$ and $V(s)$ are unimodular. It is easy to see that $\mathfrak{B} = \{ w \mid \exists a \text{ with } U(\sigma)R(\sigma)w = \Delta(\sigma)V^{-1}(\sigma)a \} = \{ w \mid \exists a \text{ with } U(\sigma)R(\sigma)w = \Delta(\sigma)a \}$.

Applying Lemma 1, the theorem follows immediately. ■

6. MOVING-AVERAGE REPRESENTATIONS OF ELEMENTS FROM $\underline{\mathfrak{L}}^q$

DEFINITION. An element $\mathfrak{B} \in \underline{\mathfrak{L}}^q$ is said to have a *moving-average* (MA) representation if $\exists M(s) \in \mathbb{R}^{q \times p}[s, s^{-1}]$ such that $\mathfrak{B} = \{ w \mid \exists a \text{ with } w = M(\sigma)a \}$.

From the previous part we already know that $\mathfrak{B} \in \underline{\mathfrak{L}}^q$.

THEOREM [2]. *Let $R(s) \in \mathbb{R}^{g \times q}[s, s^{-1}]$ with rank g and $\mathfrak{B} = \{ w \mid R(\sigma)w = 0 \}$. Then $\mathfrak{B}$ admits an MA representation iff* rank $R(c) = g$ $\forall 0 \neq c \in \mathbb{C}$.

In order to prove this result we need a lemma.

LEMMA 3 (See [9]). *Let $M(s) \in \mathbb{R}^{q \times p}[s, s^{-1}]$ be such that* rank $M(c) = p$ $\forall 0 \neq c \in \mathbb{C}$. *Then there is a matrix $A(s) \in \mathbb{R}^{q \times (q-p)}[s, s^{-1}]$ such that $[M(s), A(s)]$ is unimodular.*

Proof of Theorem. We only prove one part of the theorem, as the other part of the proof follows similar lines.

Let $\mathfrak{B} = \{ w \mid w = M(\sigma)a \}$. Because of Lemma 1, we can take without loss of generality $M(s) \in \mathbb{R}^{q \times p}[s, s^{-1}]$ such that rank $M(s) = p$ $\forall 0 \neq c \in \mathbb{C}$. Take, according to Lemma 3, $A(s)$ such that $[M(s), A(s)] = U(s)$ is unimodular. Then

$$\mathfrak{B} = \left\{ w \mid w = (M(\sigma), A(\sigma)) \begin{pmatrix} a \\ 0 \end{pmatrix} \right\} = \left\{ w \mid U^{-1}(\sigma) w = (I) \begin{pmatrix} a \\ 0 \end{pmatrix} \right\}.$$

Let $U^{-1}(s) = \operatorname{col}(\hat{R}(s), R(s))$; then $\mathfrak{B} = \{ w \mid \exists a \text{ with } \hat{R}(\sigma)w = a, R(\sigma)w = 0 \} = \{ w \mid R(\sigma)w = 0 \}$.

7. CONTROLLABILITY AND MOVING AVERAGE

NOTATION. $\mathfrak{B}^+ := \{w^+ \in (\mathbb{R}^q)^{\mathbb{Z}_+} \mid \exists w^- \in (\mathbb{R}^q)^{\mathbb{Z}_-}$ such that $w^- \cdot w^+ \in \mathfrak{B}\}$.

$\mathfrak{B}^- := \{w^- \in (\mathbb{R}^q)^{\mathbb{Z}_-} \mid \exists w^+ \in (\mathbb{R}^q)^{\mathbb{Z}_+}$ such that $w^- \cdot w^+ \in \mathfrak{B}\}$.

Hence $\mathfrak{B}^+$ is the *future* of $\mathfrak{B}$, and $\mathfrak{B}^-$ is the *past* of $\mathfrak{B}$.

DEFINITION [2]. $\mathfrak{B} \in \underline{\mathfrak{L}}^q$ is called *controllable* if $\forall w^- \in \mathfrak{B}^-$, $\forall w^+ \in \mathfrak{B}^+$ $\exists n \in \mathbb{Z}$, $\exists x \in (\mathbb{R}^q)^n$ such that $w^- \cdot x \cdot w^+ \in \mathfrak{B}$. So a system $\mathfrak{B}$ is precisely controllable if every past can be connected to every future in a finite number of steps such that the resulting trajectory is an element of $\mathfrak{B}$.

THEOREM [2]. *$\mathfrak{B} \in \underline{\mathfrak{L}}^q$ is controllable iff $\mathfrak{B}$ admits an MA representation.*

Proof. Suppose $\mathfrak{B} = \{w \mid w = M(\sigma)a\}$.

Suppose a is such that $a(t) = 0$, t sufficiently large; then $(M(\sigma)a)(t) = 0$ for t sufficiently large. A similar statement holds true when a is such that $a(t)$ is zero for t sufficiently small. Hence an MA model $\mathfrak{B} \in \underline{\mathfrak{L}}^q$ is controllable.

Suppose now that $\mathfrak{B} \in \underline{\mathfrak{L}}^q$ is controllable. Then it is not hard to see that every $w \in \mathfrak{B}$ is the limit (in $\mathfrak{L}^q$) of points $w_i \in \mathfrak{B}$ with $w^i(t) \neq 0$ finitely often. Define $\hat{\mathfrak{B}} := \{w \in \mathfrak{B} \mid \exists t_-(w), \exists t_+(w) \in \mathbb{Z}$ such that $w(t) = 0\ \forall t < t_-(w), \forall t > t_+(w)\}$. Then $\mathfrak{B}$ can be considered as a submodule of $\mathfrak{L}_c^q(d)$; hence $\hat{\mathfrak{B}} = \{w \mid w = M(\sigma)a^*$, a^* with compact support$\}$ and $\hat{\mathfrak{B}}$ (= closure in $\mathfrak{L}^q$) $= \mathfrak{B}$; hence $\mathfrak{B} = \{w \mid w = M(\sigma)a\}$, and this completes the proof. ■

8. AUTONOMOUS ELEMENTS IN $\underline{\mathfrak{L}}^q$

DEFINITION. $\mathfrak{B} \in \underline{\mathfrak{L}}^q$ is *autonomous* if it is a finite-dimensional linear subspace of $(\mathbb{R}^q)^{\mathbb{Z}}$.

THEOREM [2]. *$\mathfrak{B} \in \underline{\mathfrak{L}}^q$ is autonomous iff $\exists R(s) \in \mathbb{R}^{q \times q}[s]$ with rank q such that $\mathfrak{B} = \{w \mid R(\sigma)w = 0\}$.*

Proof. Take $\mathfrak{B} \in \underline{\mathfrak{L}}^q$ such that $\mathfrak{B}$ is finite-dimensional, and take $U(s) \in \mathbb{R}^{q \times q}[s, s^{-1}]$ unimodular. Then $\{\tilde{w} \in (\mathbb{R}^q)^{\mathbb{Z}} \mid \tilde{w} = U(\sigma)w$ for some $w \in \mathfrak{B}\}$ is also finite-dimensional. Take $\mathfrak{B} = \{w \mid R(\sigma)w = 0\}$. Let $U(s)$ and $V(s)$ be unimodular and such that $\Delta(s) := U(s)R(s)V(s)$ is diagonal. Then $\mathfrak{B} = \{w \mid \Delta(\sigma)V^{-1}(\sigma)w = 0\} = V(\sigma) \cdot \{w \mid \Delta(\sigma)w = 0\}$. Take $0 \neq p(s) \in \mathbb{R}[s, s^{-1}]$, and define $\mathfrak{B}(p) := \{w \in \mathbb{R}^{\mathbf{Z}} \mid p(\sigma)w = 0\}$. It is clear that $\mathfrak{B}(p)$ is a finite-dimensional linear subspace of $\mathbb{R}^{\mathbf{Z}}$. The rest of the proof being easy, we omit it. ∎

9. THE INTERPLAY BETWEEN CONTROLLABLE AND AUTONOMOUS ELEMENTS IN $\underline{\mathfrak{L}}^q$

Suppose that $\mathfrak{B} \in \underline{\mathfrak{L}}^q$ is controllable; then $U(\sigma)\mathfrak{B}$ is controllable for every $U(s) \in \mathbb{R}^{q\times q}[s, s^{-1}]$. This is immediate when we realize that controllable elements in $\underline{\mathfrak{L}}^q$ are precisely the elements admitting an MA representation. $\mathfrak{B} \in \underline{\mathfrak{L}}^q$ autonomous implies that $U(\sigma)\mathfrak{B}$ is autonomous for every $U(s) \in \mathbb{R}^{q\times q}[s, s^{-1}]$. By means of these two easy observations we prove the following.

Theorem [1]. *Every $\mathfrak{B} \in \underline{\mathfrak{L}}^q$ can be written as the sum of an autonomous and a controllable part, i.e., there is an autonomous $\mathfrak{B}_a \in \underline{\mathfrak{L}}^q$ and a controllable $\mathfrak{B}_c \in \underline{\mathfrak{L}}^q$ such that $\mathfrak{B} = \mathfrak{B}_a + \mathfrak{B}_c$.*

Proof. Let $\mathfrak{B} = \{w \mid R(\sigma)w = 0\}$. Take unimodular matrices $U(s)$ and $V(s)$ such that $\Delta(s) := U(s)R(s)V(s)$ is diagonal. Let

$$\Delta(s) := \begin{pmatrix} p_1(s) & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & p_2(s) & 0 & \cdot & \cdot & \cdot & 0 \\ \vdots & & \ddots & & & & \vdots \\ 0 & \cdots & & p_g(s) & 0 & \cdots & 0 \end{pmatrix}.$$

Without loss of generality we may assume that $0 \neq p_i(s)$, $i = 1,2,\dots,g$. Consider $\mathfrak{B}(\Delta) := \{w \mid \Delta(\sigma)w = 0\}$. Define $\tilde{\mathfrak{B}}_a := \{w \in (\mathbb{R}^q)^{\mathbf{Z}} \mid p_i(\sigma)w^i = 0,\ i = 1,2,\dots,g,\ w^i = 0,\ i = g+1,\dots,q\}$ and $\tilde{\mathfrak{B}}_c := \{w \in (\mathbb{R}^q)^{\mathbf{Z}} \mid w^i = 0,\ i = 1,2,\dots,g,\ w^i$ is arbitrary, $i = g+1,\dots,q\}$. It is evident that $\tilde{\mathfrak{B}}_a$ is autonomous and that $\tilde{\mathfrak{B}}_c$ is controllable and $\mathfrak{B}(\Delta) = \tilde{\mathfrak{B}}_c + \tilde{\mathfrak{B}}_a$. Now $\mathfrak{B} = \{w \mid \Delta(\sigma)V^{-1}(\sigma)w = 0\} = V(\sigma)\mathfrak{B}(\Delta) = V(\sigma)\tilde{\mathfrak{B}}_c + V(\sigma)\tilde{\mathfrak{B}}_a$, and we are done with the proof. ∎

10. CONTROLLABILITY REVISITED

In this section we will show that it makes sense to speak of *the* controllable part of a set $\mathfrak{B} \in \underline{\mathfrak{L}}^q$. The proof of Theorem 9 showed also that, with $\mathfrak{B} = \{w \mid R(\sigma)w = 0\}$ where rank $R(s) = g$, one can choose $\mathfrak{B}_c$ such that $g(\mathfrak{B}_c) = g$. [Actually we found an MA representation of $\mathfrak{B}_c$, say $\mathfrak{B}_c = \{w \mid w = M(\sigma)a\}$, where the column rank of $M(s)$ is precisely $q - g$. Similarly to the proof of Theorem 4, we then could construct an AR representation, say $\mathfrak{B}_c = \{w \mid R_c(\sigma)w = 0\}$, where $R_c(s)$ has rank g.]

THEOREM [2]. *Let $\mathfrak{B} \in \underline{\mathfrak{L}}^q$ be such that $g(\mathfrak{B}) = g$. Let $\mathfrak{B}_c^1$ and $\mathfrak{B}_c^2$ both be controllable elements in $\underline{\mathfrak{L}}^q$ with $g(\mathfrak{B}_c^1) = g(\mathfrak{B}_c^2) = g$ and such that $\mathfrak{B}_c^i \subseteq \mathfrak{B}$, $i = 1,2$. Then $\mathfrak{B}_c^1 = \mathfrak{B}_c^2$.*

Proof. Let $\mathfrak{B} = \{w \mid R(\sigma)w = 0\}$ with $R(s) \in \mathbb{R}^{g \times q}[s, s^{-1}]$. Let $\mathfrak{B}_c^i = \{w \mid R_i(\sigma)w = 0\}$ with $R_i(s) \in \mathbb{R}^{g \times q}[s, s^{-1}]$. By assumption we have $\operatorname{rank} R(s) = \operatorname{rank} R_1(s) = \operatorname{rank} R_2(s)$ (rank = row rank). As $\mathfrak{B}_c^i \subseteq \mathfrak{B}$ it follows that $\mathfrak{B}^\perp \subseteq (\mathfrak{B}_c^i)^\perp$, $i = 1,2$. From Theorem 2 it follows that $\mathfrak{B}^\perp = \{w^* \mid w^* = R^T(\sigma^{-1})\xi^*\}$, and similar equalities hold for $(\mathfrak{B}_c^i)^\perp$, $i = 1,2$.

Hence $\exists V_1(s) \in \mathbb{R}^{g \times g}[s, s^{-1}]$, $\exists V_2(s) \in \mathbb{R}^{g \times g}[s, s^{-1}]$ with row rank g such that $R(s) = V_1(s)R_1(s) = V_2(s)R_2(s)$. Now we use the controllability of $\mathfrak{B}_c^i$. We know that $\operatorname{rank} R_1(c) = \operatorname{rank} R_2(c) = g \quad \forall 0 \neq c \in \mathbb{C}$. Hence $V_1(s)^{-1}V_2(s)$ is unimodular, where $V_1(s)^{-1}$ has to be considered over the *rational* functions. But this implies that $\mathfrak{B}_c^1 = \mathfrak{B}_c^2$, and this had to be proved. ■

DEFINITION. Let $\mathfrak{B} \in \underline{\mathfrak{L}}^q$ with $g(\mathfrak{B}) = g$. Then the *controllable part* of $\mathfrak{B}$ is that controllable $\mathfrak{B}_c \in \underline{\mathfrak{L}}^q$ such that $\mathfrak{B}_c \subseteq \mathfrak{B}$ and such that $g(\mathfrak{B}_c) = g$.

Because of Theorem 10 this definition makes sense.

Given an AR representation of $\mathfrak{B} \in \underline{\mathfrak{L}}^q$, it is conceptually easy to calculate its controllable part. Let $\mathfrak{B} = \{w \mid R(\sigma)w = 0\}$. First remove superfluous rows in $R(s)$ so that we may assume that $R(s)$ has full row rank. Write $R(s) = K(s)R_1(s)$, where $R_1(c)$ has the same rank as $R(s)$ $\forall 0 \neq c \in \mathbb{C}$. It is not hard to show that the controllable part of $\mathfrak{B}$ is given by $\mathfrak{B}_c := \{w \mid R_1(\sigma)w = 0\}$.

11. LINEAR CONTINUOUS SHIFT-INVARIANT OPERATORS

In the previous section we already saw that $R(s) \in \mathbb{R}^{g_1 \times g_2}[s, s^{-1}]$ gives rise to a linear continuous shift-invariant operator between $\mathfrak{L}^{g_2}$ and $\mathfrak{L}^{g_1}$—to be precise, to the mapping $R(\sigma): \mathfrak{L}^{g_2} \to \mathfrak{L}^{g_1}$ defined by $w \to R(\sigma)w$.

Actually we can prove the following.

THEOREM. *Let* $L: \mathfrak{L}^{g_2} \to \mathfrak{L}^{g_1}$ *be linear, shift invariant, and continuous. Then there is a polynomial matrix* $R(s) \in \mathbb{R}^{g_1 \times g_2}[s, s^{-1}]$ *such that* $Lw = R(\sigma)w \ \forall w \in (\mathbb{R}^{g_2})^{\mathbb{Z}}$.

Proof. It is not hard to see that the proof can be reduced to the case $g_1 = g_2 = 1$. Hence we only need to prove that, with $e(t) = 0 \ \forall t \neq 0$ and $e(0) = 1$, we have $Le = a$, where $a(t) = 0$ for $|t|$ sufficiently large.

Suppose to the contrary that, with $I := \{t \in \mathbb{Z} \mid a(t) \neq 0\}$, $|I| = \infty$. Define a sequence $\{\underline{\tilde{e}}_i, \ i \in \mathbb{Z}\}$ as follows: $\underline{\tilde{e}}_0 = e$, $\underline{\tilde{e}}_i = 0 \ \forall i \notin I$, $\underline{\tilde{e}}_i = a_i^{-1}\sigma^i \underline{e}$ $\forall i \in I$. Define further $\{\underline{\hat{e}}_i, \ i \in \mathbb{Z}_+\}$ as follows: $\underline{\hat{e}}_0 = e$; $\underline{\hat{e}}_i := \sum_{j=-i}^{i} \underline{\tilde{e}}_j$, $i \geqslant 1$. By construction $\hat{e}_i \to_{i \to \infty} l$ for some $l \in \mathfrak{L}^1$. At the same time we have $(L\underline{\hat{e}}_i)(0) \to_{i \to \infty} \infty$. As L is supposed to be continuous, this cannot be; hence the proof now follows easily. ■

12. A SYSTEMS-THEORETIC INTERPRETATION OF THE SMITH FORM

DEFINITION. $\mathfrak{B}_1, \mathfrak{B}_2 \in \underline{\mathfrak{L}}^q$ are said to be *shift-equivalent* if $\exists L_1, L_2$ where $L_i: \mathfrak{L}^q \to \mathfrak{L}^q$ is a linear, continuous, shift-invariant operator, $i = 1,2$, such that $L_1 w \in \mathfrak{B}_2 \Leftrightarrow w \in \mathfrak{B}_1$ and $L_2 w \in \mathfrak{B}_1 \Leftrightarrow w \in \mathfrak{B}_2$. In this case we write $\mathfrak{B}_1 \stackrel{\sigma}{=} \mathfrak{B}_2$.

THEOREM. *Let* $\mathfrak{B}_i \in \underline{\mathfrak{L}}^q$, $i = 1,2$, *be given by* $\mathfrak{B}_i = \{w \mid R_i(\sigma)w = 0\}$, *where* $R_i(s) \in \mathbb{R}^{g_1 \times q}[s, s^{-1}]$ *and* R_1 *and* R_2 *both have full row rank. Then* $\mathfrak{B}_1 \stackrel{\sigma}{=} \mathfrak{B}_2$ *iff* R_1 *and* R_2 *have the same Smith form.*

Proof.

(1) Equal Smith forms imply of course $\mathfrak{B}_1 \stackrel{\sigma}{=} \mathfrak{B}_2$.

(2) The proof goes by using the Smith form and as it is tedious it is omitted. ■

The following corollary is immediate.

COROLLARY 1. $\mathfrak{B}_1 \stackrel{\sigma}{=} \mathfrak{B}_2 \ \Leftrightarrow \ \exists$ *unimodular* $U(s) \in \mathbb{R}^{q \times q}[s, s^{-1}]$ *such that* $\mathfrak{B}_2 = U(\sigma)\mathfrak{B}_1$.

A slight generalization of this corollary goes as follows.

COROLLARY 2. *Let* $\mathfrak{B} = \{w \mid R(\sigma)w = 0\}$, *where* $R(s) \in \mathbb{R}^{g\times q}[s, s^{-1}]$. *Let* $\hat{R}(s) \in \mathbb{R}^{q\times q}[s, s^{-1}]$ *be such that* $R(s)$ *and* $\hat{R}(s)$ *are right coprime. Then* $\hat{R}(\sigma)\mathfrak{B} \stackrel{\circ}{=} \mathfrak{B}$.

Proof. As R and $\hat{R}$ are right coprime, there are matrices $A(s)$ and $B(s)$ such that

$$U := \begin{pmatrix} \hat{R} & A \\ R & B \end{pmatrix} \in \mathbb{R}^{(q+g)\times(q+g)}[s, s^{-1}]$$

is unimodular. Now

$$\hat{\mathfrak{B}} = \left\{ \hat{w} \middle| \begin{pmatrix} \hat{w} \\ 0 \end{pmatrix} = U \begin{pmatrix} w \\ 0 \end{pmatrix} \right\}.$$

As U is unimodular, it follows that $L\hat{B} = \mathfrak{B}$, where $\hat{\mathfrak{B}} := \hat{R}(\sigma)\mathfrak{B}$, for some linear, continuous, shift-invariant L. Applying Theorem 12, the proof follows immediately. ∎

13. TRANSFER MATRICES

Let $\mathfrak{B} \in \underline{\mathfrak{L}}^q$ be given by $\mathfrak{B} = \{w \mid R(\sigma)w = 0\}$ for some $R(s) \in \mathbb{R}^{g\times q}[s, s^{-1}]$ such that rank $R = g = g(\mathfrak{B})$. Suppose we can partition w and R so that $R = (P, -Q)$, $w = (w_1, w_2)$, and so that $\det P(s) \neq 0$. Then we call $P^{-1}Q$ a *transfer matrix* of $(\mathfrak{B}, \pi)$, where π is the permutation of w such that $\pi w = (w_1, w_2)$. Now suppose that $\mathfrak{B} = \{w \mid \hat{R}(\sigma)w = 0\}$, where $\hat{R}$ is also an element of $\mathbb{R}^{g\times q}[s, s^{-1}]$. Then it follows from Theorem 2 that $R = U\hat{R}$ for some unimodular $U \in \mathbb{R}^{g\times q}[s, s^{-1}]$. Hence the notation of transfer matrix of $(\mathfrak{B}, \pi)$ is well defined. Suppose now in addition that $\mathfrak{B}$ is controllable; hence $\exists M_1, M_2$ such that $\mathfrak{B} = \{w \mid w_1 = M_1(\sigma)a,\ w_2 = M_2(\sigma)a$ for some trajectory $a\}$, where

$$\operatorname{rank}\begin{pmatrix} M_1(c) \\ M_2(c) \end{pmatrix}$$

is constant for all $0 \neq c \in \mathbb{C}$ and equal to the number of columns of $\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$ and to the number of rows of M_2; hence M_2 is a square matrix. As w_2 can be any trajectory, we hence have that M_2 is invertible over the rational functions (see also Section 5).

THEOREM. *In the notation given above we have that* $P^{-1}Q = M_1 M_2^{-1}$.

Proof. As

$$\operatorname{rank}\begin{pmatrix} M_1(c) \\ M_2(c) \end{pmatrix}$$

is constant $\forall 0 \neq c \in \mathbb{C}$ and equal to the number of columns in $\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$, there are, according to Lemma 3, matrices $A(s)$ and $B(s)$ such that

$$U := \begin{pmatrix} M_1(s) & A(s) \\ M_2(s) & B(s) \end{pmatrix}$$

is unimodular. Now

$$\mathfrak{B} = \left\{ w = (w_1, w_2) \,\middle|\, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} M_1(\sigma) & A(\sigma) \\ M_2(\sigma) & B(\sigma) \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} \right\}.$$

Let us denote the inverse of U by

$$\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix};$$

then

$$\mathfrak{B} = \left\{ w = (w_1, w_2) \,\middle|\, \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} \right\}$$

$$= \{ w \mid T_3 w_1 + T_4 w_2 = 0 \}.$$

It is not hard to prove that the number of columns of $\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$ is $q - g$; hence T_3 has precisely g rows. As U is unimodular and M_2 is invertible, it follows that

T_3 is invertible over the rational functions. Again applying Theorem 4, it follows that $-T_3^{-1}T_4 = P^{-1}Q$. We have $M_2 \in \mathbb{R}^{(q-g)\times(q-g)}[s, s^{-1}]$, and as $M_1 = T_3^{-1}T_4M_2$, it follows that $P^{-1}Q = M_1M_2^{-1}$. ∎

So far we have not spoken about inputs and outputs, neither have we mentioned input-state-output descriptions of elements $\mathfrak{B} \in \underline{\mathfrak{L}}^q$. Despite this fact, we were able to prove interesting results with a small toolbox. In the sequel we will broaden our viewpoint considerably by introducing the notions of input, output, and state-space realization. Then we will come back to AR and MA representation of elements $\mathfrak{B} \in \mathfrak{L}^q$.

14. STATE-SPACE REALIZATION

In this section we give a brief treatment of state-space realizations of elements $\mathfrak{B} \in \underline{\mathfrak{L}}^q$. For an extensive treatment the reader is referred to [2]. For the purposes of this paper we use definitions far less general than those given in [2].

Definition. Let $\mathfrak{B} \in \underline{\mathfrak{L}}^q$ be given. Then $\mathfrak{B}_s \in \underline{\mathfrak{L}}^{q+n}$ is called a *state-space realization of* $\mathfrak{B}$ if

(1) $\{w \mid \exists x \text{ with } (w,x) \in \mathfrak{B}_s\} = \mathfrak{B}$;
(2) $\mathfrak{B}_s$ obeys the *axiom of state*, that is,

$$(w^i, x^i) \in \mathfrak{B}_s, \quad i = 1,2, \quad x^1(0) = x^2(0) \quad \Rightarrow \quad (w,x) \in \mathfrak{B},$$

where $(w,x)(t) = (w^1, x^1)(t)\ \forall t < 0$ and $(w,x)(t) = (w^2, x^2)(t)\ \forall t \geqslant 0$.

It is not hard to prove that $\forall \mathfrak{B} \in \underline{\mathfrak{L}}^q\ \exists \hat{n}(\mathfrak{B}) \in \mathbb{Z}_+$ such that $\mathfrak{B}_s \in \underline{\mathfrak{L}}^{q+\hat{n}(\mathfrak{B})}$ is a state-space realization of $\mathfrak{B}$.

Definition. Let $\mathfrak{B} \in \underline{\mathfrak{L}}^q$. Then $n(\mathfrak{B}) = \min\{\hat{n}(\mathfrak{B}) \mid \exists \mathfrak{B}_s \in \mathfrak{L}^{q+\hat{n}(\mathfrak{B})}$ realizing $\mathfrak{B}\}$.

One can prove the following (see [2]): Let $\mathfrak{B} \in \underline{\mathfrak{L}}^q$ be given, and let $\mathfrak{B}_s^i \in \underline{\mathfrak{L}}^{q+n(\mathfrak{B})}$ realize $\mathfrak{B}$, $i = 1,2$. Then there is a linear bijection $S: \mathbb{R}^{n(\mathfrak{B})} \to \mathbb{R}^{n(\mathfrak{B})}$ such that $(w,x) \in \mathfrak{B}_s^1 \Leftrightarrow (w, Sx) \in \mathfrak{B}_s^2$.

Definition (See [2]). Let $\mathfrak{B}_s$ be a linear state-space system. Then we call it *past-induced* iff $\{(w,x) \in \mathfrak{B}_s,\ w(t) = 0\ \forall t < 0\} \Rightarrow \{x(0) = 0\}$. We call it *future-induced* iff $\{(w,x) \in \mathfrak{B}_s,\ w(t) = 0\ \forall t \geqslant 0\} \Rightarrow \{x(0) = 0\}$. Let $\mathfrak{B}_s \in \underline{\mathfrak{L}}^{q+n}$. Then we call $\mathfrak{B}_s$ *state-trim* iff $\{x \in \mathbb{R}^n \mid \exists (w,x) \in \mathfrak{B}_s$ with $x(0) = x\} = \mathbb{R}^n$.

One can prove the following (see [2]).

THEOREM. *Let* $\mathfrak{B}_s \in \underline{\mathfrak{L}}^{q+n}$ *realize* $\mathfrak{B}$. *Then* $n = n(\mathfrak{B})$ *iff* $\mathfrak{B}_s$ *is future-induced, past-induced, and state-trim.*

15. INPUT-STATE-OUTPUT REALIZATION

The most familiar state-space realization of elements $\mathfrak{B} \in \underline{\mathfrak{L}}^q$ is the so-called *input-state-output* (iso) *realization.*

DEFINITION. $\mathfrak{B}_s$ realizing $\mathfrak{B} \in \underline{\mathfrak{L}}^q$ is said to be *in iso form* if there is a permutation π of the variables w, $\pi w = (u, y)$, such that $\mathfrak{B}_s = \{(u, y, x) \mid \sigma x = Ax + Bx,\ y = Cx + Du\}$, where $A: \mathbb{R}^n \to \mathbb{R}^n$, $B: \mathbb{R}^m \to \mathbb{R}^n$, $C: \mathbb{R}^n \to \mathbb{R}^p$, $D: \mathbb{R}^m \to \mathbb{R}^p$ are all linear mappings, and where it is understood that $(Mv)(t) := Mv(t)$ for any linear mapping M and any trajectory v. The variables in u are called *inputs* and in y *outputs*. As an application of the ideas "state-trim," "past-induced," and "future-induced" we will prove the following well-known theorem.

THEOREM [2]. *Let* $\mathfrak{B}_s \in \underline{\mathfrak{L}}^{q+n}$ *be given in iso form, and realizing* $\mathfrak{B}$. *Then* $n = n(\mathfrak{B})$ *iff*

(1) $A\mathbb{R}^n + B\mathbb{R}^m = \mathbb{R}^n$,

(2) $\begin{pmatrix} \lambda I - A \\ C \end{pmatrix}$ *has rank* n $\forall \lambda \in \mathbb{C}$.

Proof. (1) is necessary for minimality. For suppose that $A\mathbb{R}^n + B\mathbb{R}^m \neq \mathbb{R}^n$. Then a nontrivial linear combination of rows of the matrix (A, B) would give a zero row, and hence $\sigma x = Ax + Bu$ would imply restrictions upon the coordinates of x, and hence $\mathbb{B}_s$ would not be state-trim.

We can always write

$$\begin{pmatrix} sI - A \\ C \end{pmatrix} = M(s)Q(s),$$

where $M(s) \in \mathbb{R}^{(m+n)\times n}[s, s^{-1}]$ has rank n $\forall c \in \mathbb{C}$. Suppose $\det Q(s) \neq d$ for some nonzero constant $d \in \mathbb{R}$. Then either $\mathfrak{B}_s$ is not future-induced or $\mathfrak{B}_s$ is not past-induced; hence (2) is also necessary for minimality to hold.

Assume now that (1) and (2) hold. Then it is easy to show that $\mathfrak{B}_s$ is minimal, and we are done with the proof. ■

16. FROM AR TO INPUT-STATE-OUTPUT

In [1] we find a procedure to construct an iso model from AR relations. It goes as follows: Let $\mathfrak{B} = \{w \mid \tilde{R}(\sigma)w = 0\}$, where $\tilde{R}(s) \in \mathbb{R}^{g' \times q}[s, s^{-1}]$. Remove first all redundant rows from $\tilde{R}$, so that $\mathfrak{B} = \{w \mid R(\sigma)w = 0\}$, where $R(s) \in \mathbb{R}^{g \times q}[s \cdot s^{-1}]$ has row rank g. Find a unimodular matrix $U(s)$ such that $\hat{R}(s) := U(s)R(s)$ is bilaterally row proper. Let $\hat{A}(s) = \operatorname{diag}(s^{\delta_1}, \dots, s^{\delta_g})\hat{R}_h + \hat{R}_{\text{rest}}$ (see Section 2). By definition $\hat{R}_h \in \mathbb{R}^{g \times q}$ has rank g. Pick g independent columns from $\hat{R}_h$, and let the corresponding subvector of w be equal to y by definition. The rest of w will be called u. Based on data in $\hat{R}(s)$, one now constructs A, B, C, and D such that $\sigma x = Ax + Bu$, $y = Cx + Du$ realizes $\mathfrak{B}$ in a minimal way and such that $n(\mathfrak{B}) = \sum_{i=1}^{g} \delta_i$.

That means that it is rather easy in principle to determine $n(\mathfrak{B})$ and possible partitionings of w into inputs and outputs. Actually one can prove that *all* partitionings into inputs and outputs can be found this way.

17. TRANSFER FUNCTIONS, INPUTS, AND OUTPUTS

In [2] the reader can find very general definitions of the notions of *input* and *output*. In the sequel we will not *explicitly* use these definitions. Instead we are satisfied with inputs and outputs in an input-state-output setting. In this section, however, we will make a connection with so-called *proper* transfer functions. We recall the following: Let $\mathfrak{B}$ be given by autoregressive equations, i.e., $\mathfrak{B} = \{w \mid R(\sigma)w = 0\}$ with $R(s) \in \mathbb{R}^{g \times q}[s, s^{-1}]$. We assume that $\operatorname{rank} R = g$. Let us partition R as $[P, -Q]$ such that $P^{-1}Q$ exists as a rational matrix. The expression $P^{-1}Q$ is by definition a so-called *transfer function*, or *transfer matrix*. We call it *proper* when $\lim_{|c| \to \infty} P^{-1}(c)Q(c)$ is finite for all sequences $\{c_i\} \subseteq \mathbb{C}$ such that $|c_i| \to \infty$. Let $P^{-1}Q$ be proper. Let the corresponding partitioning of w be $\pi w = (y, u)$, i.e., $P(\sigma)y = Q(\sigma)u$. In the sequel we will show that y can be considered as output, and hence u as input and that $\lim_{|c| \to \infty} P^{-1}(c)Q(c) = L$ for some matrix L, not dependent upon a *particular* sequence $\{c_i \mid |c_i| \to \infty\}$.

Take a unimodular matrix $U(s) \in \mathbb{R}^{g \times g}[s, s^{-1}]$ such that with $\hat{P}(s) = U(s)P(s)$ the matrix $\hat{P}(s)$ is *row-proper*, i.e.,

$$\hat{P}(s) = \begin{pmatrix} s^{\delta_1} & & 0 \\ & \ddots & \\ 0 & & s^{\delta_g} \end{pmatrix} \hat{P}_h + \hat{P}_{\text{rest}},$$

where the highest degree (in $\mathbb{R}[s]$) terms are collected in $\operatorname{diag}(s^{\delta_1}, \dots, s^{\delta_g})\hat{P}_h$ and where $\hat{P}_h \in \mathbb{R}^{g \times g}$ is nonsingular. It is not hard to show, with $\hat{Q} = UQ$, that $(\hat{P}, -\hat{Q})$ is row-proper with leading coefficient matrix $(\hat{P}_h, -\hat{Q}_h)$. Suppose this were not the case; then $\hat{P}^{-1}\hat{Q}$ would not be proper, but $\hat{P}^{-1}\hat{Q} = P^{-1}Q$, and a contradiction arises. Starting from this row-proper representation, we can, by means of unimodular transformations, transform $(\hat{P}, -\hat{Q})$ into a *bilaterally row-proper* matrix $(\tilde{P}, -\tilde{Q}) = \tilde{R}$ such that all the columns corresponding to y are independent in $\tilde{R}_h$, the matrix of leading coefficients of $\tilde{R}$. Keeping in mind the previous paragraph, it now follows that y can indeed be considered as output, and hence u as input. It is also easy to see that $\lim_{|c| \to \infty} P^{-1}Q$ is independent of the sequence $\{c_i\}$ converging to infinity.

It is evident that, starting from a bilaterally row-proper R with $\mathfrak{B} = \{w \mid R(\sigma)w = 0\}$, partitioning of w in (y, u) as done in the previous paragraph gives rise to a proper transfer matrix.

Now let an iso representation be given of a set $\mathfrak{B} \in \underline{\mathfrak{L}}^q$, i.e., $\mathfrak{B} = \{w \mid \exists x$ with $\sigma x = Ax + Bu,\ y = Cx + Du\}$, where, with $n = n(\mathfrak{B})$, $x \in (\mathbb{R}^n)^{\mathbf{Z}}$; hence

$$\mathfrak{B} = \left\{ w = (u, y) \,\middle|\, \exists x \text{ with } \begin{pmatrix} B & 0 \\ D & -I \end{pmatrix} \begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} \sigma I - A \\ -C \end{pmatrix} x \right\}.$$

As this is an ARMA representation of $\mathfrak{B}$, we can eliminate x by means of a unimodular matrix

$$\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$$

as follows:

$$\begin{aligned} \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} B & 0 \\ D & -I \end{pmatrix} \begin{pmatrix} u \\ y \end{pmatrix} &= \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} \sigma I - A \\ -C \end{pmatrix} x \\ &= \begin{pmatrix} I(s) \\ 0 \end{pmatrix} x, \end{aligned}$$

where $I(s) \in \mathbb{R}^{n \times n}[s, s^{-1}]$ is unimodular. Hence $\mathfrak{B} = \{w = (u, y) \mid U_{21}B + U_{22}D)u - U_{22}y = 0\}$, where U_{22} is square. As $\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$ is unimodular and $sI - A$ is invertible over the rational functions, it follows that U_{22}^{-1} exists as a rational matrix. We are now going to consider $U_{22}^{-1}(U_{21}B + U_{22}D) = U_{22}^{-1}U_{21}B + D$. We also know that $U_{22}^{-1}U_{21}(\sigma I - A) + C = 0$; hence $U_{22}^{-1}U_{21}B + D = -C(\sigma I - A)^{-1}B + D$.

Summarizing, we have proved the following result.

THEOREM. *Let $\mathfrak{B} \in \underline{\mathfrak{L}}^q$ be given and $\pi w = (y, u)$ also be given. Then there is an input-state-output realization of $\mathfrak{B}$ with $\sigma w = Ax + Bu$, $(y = Cx + Du)$ if and only if, with $\mathfrak{B} = \{w \mid R(\sigma)w = 0\}$ with $R(s) \in \mathbb{R}^{g \times q}[s, s^{-1}]$ such that R has rank g, the partitioning of R corresponding to $\pi w = (y, u)$ is such that, with $R = (P, -Q)$, P^{-1} exists and $P^{-1}Q$ is a proper transfer matrix.*

Hence there is a one-one relation between iso realizations and partitionings of w corresponding with proper transfer matrices. Of course, this result is not new, but the way it fits into the polynomial framework developed by J. C. Willems in a number of papers is nice and instructive.

18. AR REPRESENTATIONS REVISITED

In this section we will again demonstrate the usefulness of bilaterally row-proper polynomial matrices. Let $\mathfrak{B} \in \underline{\mathfrak{L}}^q$ be given. Let us denote the set $\{(x_0, x_1, \dots, x_T) \in \mathbb{R}^{(T+1)q} \mid \exists w \in \mathfrak{B}$ with $(w(0), \dots, w(T)) = (x_0, \dots, x_T)\}$ by $\mathfrak{B}[0, T]$. That means that $\mathfrak{B}[0, T]$ is the projection of $\mathfrak{B}$ on the time interval $[0, T]$. Let $\mathfrak{B} = \{w \mid R(\sigma)w = 0\}$, where $R(s) \in \mathbb{R}^{g \times q}[s]$ is bilaterally row-proper. Let $R(s) := \operatorname{col}(r_1(s), r_2(s), \dots, r_g(s))$, and let $r_i(s) := s^{\delta_i}r_{i,\delta_i} + s^{\delta_i - 1}r_{i,\delta_i - 1} + \cdots + s^0 r_{i,0}$, with $r_{i,j} \in \mathbb{R}^q$, for $i = 1, 2, \dots, g$ and $j = 0, 1, \dots, \delta_i$, and $\delta_i =: \deg r_i(s)$. Take $T \in \mathbb{Z}_+$ such that $T \geqslant \max \delta_i$, and define $Q := \{(x_0, x_1, \dots, x_T) \in \mathbb{R}^{(T+1)q} \mid r_{i,0}x_0 + r_{i,1}x_1 + \cdots + r_{i,\delta_i}x_{\delta_i} = 0, \dots, r_{i,0}x_{T-\delta_i} + \cdots + r_{i,\delta_i}x_T = 0\}$.

THEOREM. $\mathfrak{B}[0, T] = Q$.

Proof. Easy. ■

COROLLARY. *Theorem* 4.2.

Proof. Without loss of generality, take $R_1(s)$ and $R_2(s)$ bilaterally row-proper. $\mathfrak{B}_1 = \mathfrak{B}_2$ implies $\mathfrak{B}_1[0, T] = \mathfrak{B}_2[0, T]$ $\forall T \in \mathbb{Z}_+$. Applying Theorem 18, the result almost immediately follows. ■

19. STATE-SPACE REALIZATIONS STARTING FROM A MOVING-AVERAGE REPRESENTATION

In the foregoing we sketched a way to find an input-state-output realization starting from an AR representation of an element $\mathfrak{B} \in \underline{\mathfrak{L}}^q$. In doing this we discovered that $n(\mathfrak{B})$, the minimal dimension of the state space in a state-space realization of $\mathfrak{B}$, is equal to $\sum_{i=1}^{g} \delta_i$, where δ_i is the degree of the ith row of $R(s)$, where $R(s)$ is bilaterally row-proper and such that $\mathfrak{B} = \{ w \mid R(\sigma) w = 0 \}$.

When $\mathfrak{B}$ is controllable, it also allows for an MA representation, $\mathfrak{B} = \{ w \mid \exists a \text{ with } w = M(\sigma) a \}$, where $M(s) \in \mathbb{R}^{q \times m}[s, s^{-1}]$. In the sequel we will prove that for suitable M we have $n(\mathfrak{B}) = \sum_{i=1}^{m} \hat{\delta}_i$, where $\hat{\delta}_i$ is the degree of the ith *column* of M. Actually we will do more and also give a new procedure to construct a state-space realization starting from an MA representation of $\mathfrak{B}$.

Let $\mathfrak{B} = \{ w \mid M(\sigma) a = w \}$, where $M(s) \in \mathbb{R}^{q \times m}[s, s^{-1}]$ can be written as $M(s) = \sum_{i=-k}^{l} M_i s^i$ with $M_i \in \mathbb{R}^{q \times m}$. We consider two cases: (1) $l = 0$ and (2) $k = 0$. We can always multiply by suitable powers of s so that (1) or (2) holds. In both cases we assume, without loss of generality, that M_0 does not contain zero columns.

(1): $l = 0$; hence $M(s) = M_{-k} s^{-k} + \cdots + M_0$. Let

$$\delta_i := \text{degree of } i\text{th column of } M(s),$$

$$M^i_{-j} := i\text{th column of } M_{-j},$$

$$A_i := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{\delta_i \times \delta_i}, \qquad B_i := \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^{\delta_i},$$

$$A := \operatorname{diag}(A_1, \ldots, A_m), \qquad B := \operatorname{diag}(B_1, \ldots, B_m),$$

$$C := \left(M^1_{-\delta_1}, M^1_{-\delta_1+1}, \ldots, M^1_{-1} M^2_{-\delta_2}, \ldots, M^2_{-1}, \ldots, M^m_{-\delta_m}, \ldots, M^m_{-1} \right) \in \mathbb{R}^{q \times (\Sigma \delta_i)},$$

$$D := M_0.$$

It is not hard to see that

$$\mathfrak{B} = \{ w \mid \exists x, \exists v \text{ such that } \sigma x = Ax + Bv,\ w = Cx + Dv \},$$

and $\{(w, x) \mid \exists v \text{ with } \sigma x = Ax + Bv,\ w = Cx + Dv\}$ is easily seen to be a state-space model. By construction the dimension of the state space is equal to $\sum \delta_i$. Here we made the following identification: $x_i(t) \cong (a_i(t-\delta_i), \ldots, a_i(t-1))$. The variables v are called *driving variables.*

(2): $k = 0$; hence $M(s) = M_0 + M_1 s + \cdots + M_l s^l$. Again

$$\delta_i := \text{degree of } i\text{th column of } M(s),$$

$$\hat{C} := \left(M^1_{\delta_1}, M^1_{\delta_1 - 1}, \ldots, M^1_1, M^2_{\delta_2}, \ldots, M^2_1, \ldots, M^m_{\delta_m}, \ldots, M^m_1\right) \in \mathbb{R}^{q \times (\Sigma \delta_i)}.$$

A, B, and D are defined as above. Now we have $\mathfrak{B} = \{w \mid \exists x, v \text{ such that } \sigma^{-1}x = Ax + Bv,\ w = \hat{C}x + Dv\}$, and again the dimension of the state space is equal to $\sum \delta_i$.

Notice that under (1) we have a model in σ, whereas under (2) the model contains σ_i^{-1}.

THEOREM. *In the construction above let the column rank of $M(s)$ be equal to m, the number of columns of $M(s)$. Then the state-space realizations constructed under* (1) *and* (2) *are minimal precisely when $M(s)$ is bilaterally column proper. The dimension of the state space is in both cases equal to $\sum \delta_i$, where δ_i is the degree of the ith column of $M(s)$.*

Proof. It is not difficult to prove that the realizations under (1) and (2) are state-trim, future-induced, and past-induced precisely when $M^T(s)$ is bilaterally row-proper. The construction shows that $n(\mathfrak{B}) = \sum \delta_i$, where $\mathfrak{B} = \{w \mid w = M(\sigma)a\}$. ∎

The realizations under (1) and (2) are so-called *Brunovsky canonical realizations*; see also [5, 10].

20. DUALITY

Let $\mathfrak{B} \in \underline{\mathfrak{L}}^q$. Then $\mathfrak{B}^\perp \subseteq W^*$ is defined by $\mathfrak{B}^\perp =: \{w^* \mid \langle w^*, \mathfrak{B}\rangle = 0\} = \{w^* \mid \langle w^*, w\rangle = 0 \ \ \forall w \in \mathfrak{B}\}$. Let us recall that $\langle w^*, w\rangle := \sum_{t \in \mathbf{Z}} \sum_{i=1}^q w_i^*(t) w_i(t) = \sum_{i=1}^q \sum_{t \in \mathbf{Z}} w_i^*(t) w_i(t) =: \sum_{i=1}^q \langle w_i, w_i^*\rangle$. We already know that $\mathfrak{B}^{\perp}$ is a finitely generated submodule of $(\mathbb{R}^q)_c^{\mathbf{Z}} := \{w \in (\mathbb{R}^q)^{\mathbf{Z}} \mid w(t) \neq 0$ finitely often$\}$; see the first paragraphs of this paper. Using reasonings similar to those in Section 2, one can prove the following.

Let $\mathfrak{B} = \{w \mid R(\sigma)w = 0\}$; *then* $\mathfrak{B}^{\perp} = \{w^* = R^T(\sigma^{-1})\xi^*\}$, *where* $R^T(\sigma^{-1})$ *is the matrix that results from transposing* $R(\sigma)$ *and replacing* σ^i *by* σ^{-i} $\forall i \in \mathbb{Z}$ *in the resulting matrix.*

Again it is understood that $\xi^*(t) = 0$ $\forall t \leqslant t_-(\xi^*)$, $\forall t \geqslant t_+(\xi^*)$, where $t_-(\xi^*)$ and $t_+(\xi^*)$ are elements from $\mathbb{Z}$. One can also prove quite easily that $\mathrm{cl}(p)\mathfrak{B}^{\perp}$, the closure of $\mathfrak{B}^{\perp}$ in the topology of pointwise convergence on $(\mathbb{R}^q)^{\mathbf{Z}}$, is equal to $\{w \mid w = R^T(\sigma^{-1})\xi\}$. Hence $\mathrm{cl}(p)\mathfrak{B}^{\perp} \in \underline{\mathfrak{L}}^q$ and is controllable.

Let $\mathfrak{B} \in \underline{\mathfrak{L}}^{q_1+q_2}$ be given by $\{w_1, w_2) \in \mathfrak{B} \mid w_i \in (\mathbb{R}^{q_i})^{\mathbf{Z}}\}$. Then $P_1\mathfrak{B} := \{w_1 \in (\mathbb{R}^{q_1})^{\mathbf{Z}} \mid \exists w_2 \in (\mathbb{R}^{q_2})^{\mathbf{Z}}$ such that $(w_1, w_2) \in \mathfrak{B}\}$. It is not hard to see that $P_1\mathfrak{B} \in \underline{\mathfrak{L}}^{q_1}$. Further, it is easy to prove that $(P_1B)^{\perp} = \{w_1^* \mid (w_1^*, 0) \in \mathfrak{B}^{\perp}\}$. By means of this result we will prove the following.

THEOREM [6, 7]. *Let* $\mathfrak{B} \in \underline{\mathfrak{L}}^q$ *be given by* $\mathfrak{B} = \{w = (u, y) \mid \exists x$ *with* $\sigma x = Ax + Bu$, $y = Cx + Du\}$, *where* A, B, C *and* D *are matrices. Then* $\mathfrak{B}^{\perp} = \{(u^*, y^*) \mid u^* = B\xi^* - Dy^*, \sigma^{-1}\xi^* = A\xi^* - Cy^*\}$.

Proof. We have

$$\mathfrak{B} = \left\{ (u, y) \,\middle|\, \exists x \text{ with } \begin{pmatrix} B & 0 & A - \sigma I \\ D & -I & C \end{pmatrix} \begin{pmatrix} u \\ y \\ x \end{pmatrix} = 0 \right\},$$

$$\hat{\mathfrak{B}} := \left\{ (u, y, x) \,\middle|\, \begin{pmatrix} B & 0 & A - \sigma I \\ D & -I & C \end{pmatrix} \begin{pmatrix} u \\ y \\ x \end{pmatrix} = 0 \right\},$$

$$\hat{B}^{\perp} = \left\{ (u^*, y^*, x^*) \,\middle|\, \begin{pmatrix} u^* \\ y^* \\ x^* \end{pmatrix} = \begin{pmatrix} B & D \\ 0 & -I \\ A - \sigma^{-1}I & C \end{pmatrix} \begin{pmatrix} \xi_1^* \\ \xi_2^* \end{pmatrix} \right\};$$

hence $\mathfrak{B}^{\perp} \cong \{(u^*, y^*, 0) \in \hat{\mathfrak{B}}^{\perp}\} = \{(u^*, y^*) \mid u^* = B\xi_1^* + D\xi_2^*,\ y^* = -\xi_2^*, 0 = (A - \sigma^{-1}I)\xi_1^* + \xi_2^*\} = \{(u^*, y^*) \mid u^* = B\xi^* - D\xi^*,\ \sigma^{-1}\xi^* = A\xi^* - Cy^*\}$. Hence $\mathrm{cl}(p)\mathfrak{B}^{\perp} = \{(\tilde{u}, \tilde{y}) \mid \exists \tilde{x}$ with $\tilde{u} = B\tilde{x} - D\tilde{y},\ \sigma^{-1}\tilde{x} = A\tilde{x} - C\tilde{y}\}$.

Notice that this is again in input-state-output form, but instead of σx we now have $\sigma^{-1}\tilde{x}$.

For further ramifications of duality defined this way the reader is referred to [6] and [7].

21. CONTROL

We start with an example. Let $\mathfrak{B}_i \in \mathfrak{L}^2$, $i = 1,2,3$, be given by $\mathfrak{B}_1 := \{(w_1, w_2) \mid (\sigma^2 + 1)w_1 + w_2 = 0\}$, $\mathfrak{B}_2 := \{(w_1, w_2) \mid w_1 + (\sigma^2 + 2)w_2 = 0\}$, $\mathfrak{B}_3 := \{(w_1, w_2) \mid (\sigma + 2)w_1 + 2w_2 = 0\}$.

The intersection of $\mathfrak{B}_1$ with $\mathfrak{B}_2$ ($\mathfrak{B}_1 \cap \mathfrak{B}_2$) can be given a control-theoretic interpretation in the sense that $\mathfrak{B}_1$ is a plant and $\mathfrak{B}_2$ its feedback controller, for the output of $\mathfrak{B}_1$ is the input of $\mathfrak{B}_2$ and vice versa. $\mathfrak{B}_1 \cap \mathfrak{B}_3$ cannot be given a nice control-theoretic interpretation. But it seems worth while to study control-theoretic problems by studying, say, intersections of elements in $\mathfrak{L}^q$. As an example we will consider the following problem:

PROBLEM. Let $\mathfrak{B}$ and $\mathfrak{B}_1$ be given such that $\mathfrak{B}, \mathfrak{B}_1 \in \mathfrak{L}^q$ and such that $\mathfrak{B} \subseteq \mathfrak{B}_1$. Study the collection of all $\mathfrak{B}_2 \in \mathfrak{L}^q$ such that $\mathfrak{B}_1 \cap \mathfrak{B}_2 = \mathfrak{B}$.

In the light of the foregoing remarks and example it is not hard to see the control-theoretic importance of this problem.

Let, for arbitrary $\tilde{\mathfrak{B}} \in \mathfrak{L}^q$, $p(\tilde{\mathfrak{B}})$ be the number of outputs of $\tilde{\mathfrak{B}}$. As seen in previous parts of this paper, $p(\tilde{\mathfrak{B}}) = g(\tilde{\mathfrak{B}})$.

Let $n(\tilde{\mathfrak{B}})$ be the dimension of the state space of a *minimal* state-space realization of $\tilde{\mathfrak{B}}$. As proved before, this number $n(\tilde{\mathfrak{B}})$ is well defined.

Let $\mathfrak{B} = \mathfrak{B}_1 \cap \mathfrak{B}_2$ with $\mathfrak{B}_i \in \mathfrak{L}^q$, $i = 1,2$. Then, using previous results in this paper, it is not hard to see that the following inequalities hold:

$$p(\mathfrak{B}) \leqslant p(\mathfrak{B}_1) + p(\mathfrak{B}_2), \qquad n(\mathfrak{B}) \leqslant n(\mathfrak{B}_1) + n(\mathfrak{B}_2).$$

THEOREM. *Let $\mathfrak{B}, \mathfrak{B}_1 \in \mathfrak{L}^q$ be given. A necessary and sufficient condition such that there is a $\mathfrak{B}_2 \in \mathfrak{L}^q$ with $\mathfrak{B}_1 \cap \mathfrak{B}_2 = \mathfrak{B}$ and such that $p(\mathfrak{B}) = p(\mathfrak{B}_1) + p(\mathfrak{B}_2)$ is the following: There is a polynomial matrix $F_1(s)$ such that* rank $F_1(c)$ *is the number of rows of F_1 over $0 \neq c \in \mathbb{C}$ such that $R_1 = F_1 R$, where by definition $\mathfrak{B} = \{w \mid R(\sigma)w = 0\}$ and $\mathfrak{B}_1 = \{w \mid R_1(\sigma)w = 0\}$.*

In this case the collection of "good" $\mathfrak{B}_2$'s is parametrized by F_2 in the following sense: $\mathfrak{B}_2$ is "good" $\Leftrightarrow$ $\exists F_2$ such that $\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$ is unimodular and $\mathfrak{B}_2 = \{w \mid R_2(\sigma)w = 0\}$ with $R_2 = F_2 R$.

Proof. Suppose that $\mathfrak{B}_i = \{w \mid R_i(\sigma)w = 0\}$ with $R_i(s) \in \mathbb{R}^{g_i \times q}[s]$ and with $g_i = p(\mathfrak{B}_i)$, and that $\mathfrak{B} = \mathfrak{B}_1 \cap \mathfrak{B}_2$. Then trivially

$$\mathfrak{B} = \{w \mid R(\sigma)w = 0\} \qquad \text{with} \quad R = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}.$$

It is evident that F_1 can be chosen properly.

Suppose now that $\mathfrak{B} = \{w \mid R(\sigma)w = 0\}$, that $\mathfrak{B}_1 = \{w \mid R_1(\sigma)w = 0\}$, and that $R_1 = F_1 R$, where rank $F_1(c)$ is the number of rows of F_1, $\forall 0 \neq x \in \mathbb{C}$. From Lemma 3 we know that there is a polynomial matrix F_2 such that $\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$ is unimodular. Define $R_2 := F_2 R$ and $\mathfrak{B}_2 := \{w \mid R_2(\sigma)w = 0\}$. The rest of the proof being trivial, we omit it. ■

As an example we consider

$$R(s) := \begin{pmatrix} s+1 & 1 \\ s+2 & 2 \end{pmatrix}, \qquad R_1 = (s+1, 1).$$

Trivially we have $R_1 = F_1 R$ with $F_1 = (1,0)$. The class of all R_2 such that $\mathfrak{B}_2 = \{w \mid R_2(\sigma)w = 0\}$ and such that $\mathfrak{B}_1 \cap \mathfrak{B}_2 = \mathfrak{B}$ and $p(\mathfrak{B}) = p(\mathfrak{B}_1) + p(\mathfrak{B}_2)$ is given by

$$\left\{ F_2 R \,\middle|\, \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \text{ is unimodular} \right\}$$

$$= \left\{ F_2 R \,\middle|\, F_2 = \left(\alpha(s), bs^k\right),\ \alpha(s) \in \mathbb{R}[s] \text{ arbitrary},\ b \neq 0 \right\}$$

$$= \left\{ R_2 \,\middle|\, R_2 = \left(\alpha(s)(s+1) + bs^k(s+2), \alpha(s) + 2bs^k\right) \right\}.$$

It is easily seen that $n(\mathfrak{B}_1) = 1 = n(\mathfrak{B})$ and $n(\mathfrak{B}_2) \geqslant 1$, and further that the output in $\mathfrak{B}_1$ is at the same time the output in $\mathfrak{B}_2$.

22. CONTROL REVISITED

Let us take $\mathfrak{B}$, $\mathfrak{B}_1$, and $\mathfrak{B}_2 \in \underline{\mathfrak{L}}^q$ such that $\mathfrak{B} = \mathfrak{B}_1 \cap \mathfrak{B}_2$ and such that $p(\mathfrak{B}) = p(\mathfrak{B}_1) + p(\mathfrak{B}_2)$. What can we say about $n(\mathfrak{B})$ versus $n(\mathfrak{B}_1) + n(\mathfrak{B}_2)$ in relation to variables being input or output? An answer to this question gives

THEOREM. *Generically $n(\mathfrak{B}) = n(\mathfrak{B}_1) + n(\mathfrak{B}_2)$ if and only if one can take the outputs of $\mathfrak{B}_1$ to be the inputs of $\mathfrak{B}_2$.*

Proof. Let $\mathfrak{B}_i = \{w \mid R_i(\sigma)w = 0\}$, with $R_i(s) \in \mathbb{R}^{g_i \times q}[s, s^{-1}]$ and $g_i = p(\mathfrak{B}_i)$, $i = 1,2$. Take R_1 and R_2 further such that both matrices are bilaterally row-reduced. Now $n(\mathfrak{B}) = n(\mathfrak{B}_1) + n(\mathfrak{B}_2)$ precisely when $\begin{pmatrix} R_1 \\ R_2 \end{pmatrix}$ is bilaterally row-reduced, and this is precisely the case when there is a partitioning of w, say $\pi w = (w_1, w_2)$, such that w_1 contains all the outputs of $\mathfrak{B}_1$ and such that w_2 contains all the outputs of $\mathfrak{B}_2$. This follows from the Laplace expansion of a determinant, see [g].

23. CONTROL FURTHER REVISITED

Another example of an approach to control problems in a polynomial setting is exemplified by the following.

Let $\mathfrak{B}$ be a controllable element in $\underline{\mathfrak{L}}^q$; hence $\mathfrak{B} = \{(u, y) \mid y = M_1, \xi, u = M_2\xi\}$, where

$$\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \in \mathbb{R}^{q \times m}[s, s^{-1}]$$

and where M_1 and M_2 can be taken to be coprime. It is understood that u is input and y is output. The linear output feedback $u = Fy + v$ leads to $\mathfrak{B}(F) := \{(v, y) \mid v = (M_2 - FM_1)\xi,\ y = M_1\xi\}$; hence study of the set $\{\tilde{M} \mid \tilde{M} = M_2 - FM_1$ for some matrix $F\}$ leads to insight, for instance, into the pole placement problem. Conceptually there is no problem in taking F to be even a polynomial matrix in σ^{-1}.

In these last three sections we have only given a glimpse of a new approach to control problems. In the future readers can expect more results along these lines.

24. A SHORT INTRODUCTION TO RELATED LITERATURE

J. C. Willems's work is part of a nice system-theoretic fabric. He, like many others, owes a lot to Kalman's mathematical description of linear dynamical systems; see for instance [11]. The work done by Rosenbrock and

Wolovich (see for instance [12] and [13]) should also be mentioned in this respect. In these books one can find applications of the theory of polynomial matrices to linear systems theory. This polynomial-rational approach is also prominent in work done by Forney [14] and Fuhrmann [15, 16]. In [16] the reader can find a nice treatment of duality in the framework of polynomial models.

It is evident that it is impossible to do justice to everyone in the system-theoretic community in a brief section like this. The book written by Kailath [5] contains a wealth of related literature and useful historical remarks. The lists of references in [1] and [2] also give a good entry into related literature.

25. FINAL REMARKS

In the foregoing we have tried to give an idea of the power of J. C. Willems's approach to linear systems theory. It appears that by using a small toolbox of linear polynomial algebra one can achieve important systems-theoretical results.

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